

# FORMALITY FOR ALGEBROID STACKS

PAUL BRESSLER, ALEXANDER GOROKHOVSKY, RYSZARD NEST,  
AND BORIS TSYGAN

ABSTRACT. We extend the formality theorem of M. Kontsevich from deformations of the structure sheaf on a manifold to deformations of gerbes.

## 1. INTRODUCTION

In the fundamental paper [11] M. Kontsevich showed that the set of equivalence classes of formal deformations the algebra of functions on a manifold is in one-to-one correspondence with the set of equivalence classes of formal Poisson structures on the manifold. This result was obtained as a corollary of the formality of the Hochschild complex of the algebra of functions on the manifold conjectured by M. Kontsevich (cf. [10]) and proven in [11]. Later proofs by a different method were given in [14] and in [5].

In this paper we extend the formality theorem of M. Kontsevich to deformations of gerbes on smooth manifolds, using the method of [5]. Let  $X$  be a smooth manifold; we denote by  $\mathcal{O}_X$  the sheaf of complex valued  $C^\infty$  functions on  $X$ . For a twisted form  $\mathcal{S}$  of  $\mathcal{O}_X$  regarded as an algebroid stack (see Section 2.5) we denote by  $[\mathcal{S}]_{dR} \in H_{dR}^3(X)$  the de Rham class of  $\mathcal{S}$ . The main result of this paper establishes an equivalence of 2-groupoid valued functors of Artin  $\mathbb{C}$ -algebras between  $\text{Def}(\mathcal{S})$  (the formal deformation theory of  $\mathcal{S}$ , see [2]) and the Deligne 2-groupoid of Maurer-Cartan elements of  $L_\infty$ -algebra of multivector fields on  $X$  twisted by a closed three-form representing  $[\mathcal{S}]_{dR}$ :

**Theorem 6.1.** *Suppose that  $\mathcal{S}$  is a twisted form of  $\mathcal{O}_X$ . Let  $H$  be a closed 3-form on  $X$  which represents  $[\mathcal{S}]_{dR} \in H_{dR}^3(X)$ . For any Artin algebra  $R$  with maximal ideal  $\mathfrak{m}_R$  there is an equivalence of 2-groupoids*

$$\text{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \text{Def}(\mathcal{S})(R)$$

*natural in  $R$ .*

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Here,  $\mathfrak{s}(\mathcal{O}_X)_H$  denotes the  $L_\infty$ -algebra of multivector fields with the trivial differential, the binary operation given by Schouten bracket, the ternary operation given by  $H$  (see 5.3) and all other operations equal to zero. As a corollary of this result we obtain that the isomorphism classes of formal deformations of  $\mathcal{S}$  are in a bijective correspondence with equivalence classes of the formal *twisted Poisson structures* defined by P. Severa and A. Weinstein in [13].

The proof of the Theorem proceeds along the following lines. As a starting point we use the construction of the Differential Graded Lie Algebra (DGLA) controlling the deformations of  $\mathcal{S}$ . This construction was obtained in [1, 2]. Next we construct a chain of  $L_\infty$ -quasi-isomorphisms between this DGLA and  $\mathfrak{s}(\mathcal{O}_X)_H$ , using the techniques of [5]. Since  $L_\infty$ -quasi-isomorphisms induce equivalences of respective Deligne groupoids, the result follows.

The paper is organized as follows. Section 2 contains the preliminary material on jets and deformations. Section 3 describes the results on the deformations of algebroid stacks. Section 4 is a short exposition of [5]. Section 5 contains the main technical result of the paper: the construction of the chain of quasi-isomorphisms mentioned above. Finally, in Section 6 the main theorem is deduced from the results of Section 5.

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## 2. PRELIMINARIES

**2.1. Notations.** Throughout this paper, unless specified otherwise,  $X$  will denote a  $C^\infty$  manifold. By  $\mathcal{O}_X$  we denote the sheaf of complex-valued  $C^\infty$  functions on  $X$ .  $\mathcal{A}_X^\bullet$  denotes the sheaf of differential forms on  $X$ , and  $\mathcal{T}_X$  the sheaf of vector fields on  $X$ . For a ring  $K$  we denote by  $K^\times$  the group of invertible elements of  $K$ .

**2.2. Jets.** Let  $\text{pr}_i: X \times X \rightarrow X$ ,  $i = 1, 2$ , denote the projection on the  $i^{\text{th}}$  factor. Let  $\Delta_X: X \rightarrow X \times X$  denote the diagonal embedding. Let  $\mathcal{I}_X := \ker(\Delta_X^*)$ .

For a locally-free  $\mathcal{O}_X$ -module of finite rank  $\mathcal{E}$  let

$$\begin{aligned} \mathcal{J}_X^k(\mathcal{E}) &:= (\text{pr}_1)_* \left( \mathcal{O}_{X \times X} / \mathcal{I}_X^{k+1} \otimes_{\text{pr}_2^{-1} \mathcal{O}_X} \text{pr}_2^{-1} \mathcal{E} \right), \\ \mathcal{J}_X^k &:= \mathcal{J}_X^k(\mathcal{O}_X). \end{aligned}$$

It is clear from the above definition that  $\mathcal{J}_X^k$  is, in a natural way, a commutative algebra and  $\mathcal{J}_X^k(\mathcal{E})$  is a  $\mathcal{J}_X^k$ -module.

Let

$$\mathbf{1}^{(k)}: \mathcal{O}_X \rightarrow \mathcal{J}_X^k$$

denote the composition

$$\mathcal{O}_X \xrightarrow{\text{pr}_1^*} (\text{pr}_1)_* \mathcal{O}_{X \times X} \rightarrow \mathcal{J}_X^k$$

In what follows, unless stated explicitly otherwise, we regard  $\mathcal{J}_X^k(\mathcal{E})$  as a  $\mathcal{O}_X$ -module via the map  $\mathbf{1}^{(k)}$ .

Let

$$j^k: \mathcal{E} \rightarrow \mathcal{J}_X^k(\mathcal{E})$$

denote the composition

$$\mathcal{E} \xrightarrow{e \mapsto 1 \otimes e} (\text{pr}_1)_* \mathcal{O}_{X \times X} \otimes_{\mathbb{C}} \mathcal{E} \rightarrow \mathcal{J}_X^k(\mathcal{E})$$

The map  $j^k$  is not  $\mathcal{O}_X$ -linear unless  $k = 0$ .

For  $0 \leq k \leq l$  the inclusion  $\mathcal{I}_X^{l+1} \rightarrow \mathcal{I}_X^{k+1}$  induces the surjective map  $\pi_{l,k}: \mathcal{J}_X^l(\mathcal{E}) \rightarrow \mathcal{J}_X^k(\mathcal{E})$ . The sheaves  $\mathcal{J}_X^k(\mathcal{E})$ ,  $k = 0, 1, \dots$  together with the maps  $\pi_{l,k}$ ,  $k \leq l$  form an inverse system. Let  $\mathcal{J}_X(\mathcal{E}) = \mathcal{J}_X^\infty(\mathcal{E}) := \varprojlim \mathcal{J}_X^k(\mathcal{E})$ . Thus,  $\mathcal{J}_X(\mathcal{E})$  carries a natural topology.

The maps  $\mathbf{1}^{(k)}$  (respectively,  $j^k$ ),  $k = 0, 1, 2, \dots$  are compatible with the projections  $\pi_{l,k}$ , i.e.  $\pi_{l,k} \circ \mathbf{1}^{(l)} = \mathbf{1}^{(k)}$  (respectively,  $\pi_{l,k} \circ j^l = j^k$ ). Let  $\mathbf{1} := \varprojlim \mathbf{1}^{(k)}$ ,  $j^\infty := \varprojlim j^k$ .

Let

$$d_1: \mathcal{O}_{X \times X} \otimes_{\text{pr}_2^{-1} \mathcal{O}_X} \text{pr}_2^{-1} \mathcal{E} \longrightarrow \text{pr}_1^{-1} \mathcal{A}_X^1 \otimes_{\text{pr}_1^{-1} \mathcal{O}_X} \mathcal{O}_{X \times X} \otimes_{\text{pr}_2^{-1} \mathcal{O}_X} \text{pr}_2^{-1} \mathcal{E}$$

denote the exterior derivative along the first factor. It satisfies

$$d_1(\mathcal{I}_X^{k+1} \otimes_{\text{pr}_2^{-1} \mathcal{O}_X} \text{pr}_2^{-1} \mathcal{E}) \subset \text{pr}_1^{-1} \mathcal{A}_X^1 \otimes_{\text{pr}_1^{-1} \mathcal{O}_X} \mathcal{I}_X^k \otimes_{\text{pr}_2^{-1} \mathcal{O}_X} \text{pr}_2^{-1} \mathcal{E}$$

for each  $k$  and, therefore, induces the map

$$d_1^{(k)}: \mathcal{J}_X^k(\mathcal{E}) \rightarrow \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}_X^{k-1}(\mathcal{E})$$

The maps  $d_1^{(k)}$  for different values of  $k$  are compatible with the maps  $\pi_{l,k}$  giving rise to the *canonical flat connection*

$$\nabla^{\text{can}}: \mathcal{J}_X(\mathcal{E}) \rightarrow \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E}) .$$

**2.3. Deligne groupoids.** In [4] P. Deligne and, independently, E. Getzler in [8] associated to a nilpotent DGLA  $\mathfrak{g}$  concentrated in degrees greater than or equal to  $-1$  the 2-groupoid, referred to as *the Deligne 2-groupoid* and denoted  $\mathrm{MC}^2(\mathfrak{g})$  in [1], [2] and below. The objects of  $\mathrm{MC}^2(\mathfrak{g})$  are the Maurer-Cartan elements of  $\mathfrak{g}$ . We refer the reader to [8] (as well as to [2]) for a detailed description. The above notion was extended and generalized by E. Getzler in [7] as follows.

To a nilpotent  $L_\infty$ -algebra  $\mathfrak{g}$  Getzler associates a (Kan) simplicial set  $\gamma_\bullet(\mathfrak{g})$  which is functorial for  $L_\infty$  morphisms. If  $\mathfrak{g}$  is concentrated in degrees greater than or equal to  $1 - l$ , then the simplicial set  $\gamma_\bullet(\mathfrak{g})$  is an  $l$ -dimensional hypergroupoid in the sense of J.W. Duskin (see [6]) by [7], Theorem 5.4.

Suppose that  $\mathfrak{g}$  is a nilpotent  $L_\infty$ -algebra concentrated in degrees greater than or equal to  $-1$ . Then, according to [6], Theorem 8.6 the simplicial set  $\gamma_\bullet(\mathfrak{g})$  is the nerve of a bigroupoid, or, a 2-groupoid in our terminology. If  $\mathfrak{g}$  is a DGLA concentrated in degrees greater than or equal to  $-1$  this 2-groupoid coincides with  $\mathrm{MC}^2(\mathfrak{g})$  of Deligne and Getzler alluded to earlier. We extend our notation to the more general setting of nilpotent  $L_\infty$ -algebras as above and denote by  $\mathrm{MC}^2(\mathfrak{g})$  the 2-groupoid furnished by [6], Theorem 8.6.

For an  $L_\infty$ -algebra  $\mathfrak{g}$  and a nilpotent commutative algebra  $\mathfrak{m}$  the  $L_\infty$ -algebra  $\mathfrak{g} \otimes \mathfrak{m}$  is nilpotent, hence the simplicial set  $\gamma_\bullet(\mathfrak{g} \otimes \mathfrak{m})$  is defined and enjoys the following homotopy invariance property ([7], Proposition 4.9, Corollary 5.11):

**Theorem 2.1.** *Suppose that  $f: \mathfrak{g} \rightarrow \mathfrak{h}$  is a quasi-isomorphism of  $L_\infty$  algebras and let  $\mathfrak{m}$  be a nilpotent commutative algebra. Then the induced map*

$$\gamma_\bullet(f \otimes \mathrm{Id}): \gamma_\bullet(\mathfrak{g} \otimes \mathfrak{m}) \rightarrow \gamma_\bullet(\mathfrak{h} \otimes \mathfrak{m})$$

*is a homotopy equivalence.*

**2.4. Algebroid stacks.** Here we give a very brief overview, referring the reader to [3, 9] for the details. Let  $k$  be a field of characteristic zero, and let  $R$  be a commutative  $k$ -algebra.

**Definition 2.2.** A stack in  $R$ -linear categories  $\mathcal{C}$  on  $X$  is an  *$R$ -algebroid stack* if it is locally nonempty and locally connected, i.e. satisfies

- (1) any point  $x \in X$  has a neighborhood  $U$  such that  $\mathcal{C}(U)$  is nonempty;
- (2) for any  $U \subseteq X$ ,  $x \in U$ ,  $A, B \in \mathcal{C}(U)$  there exists a neighborhood  $V \subseteq U$  of  $x$  and an isomorphism  $A|_V \cong B|_V$ .

For a prestack  $\mathcal{C}$  we denote by  $\tilde{\mathcal{C}}$  the associated stack.

For a category  $C$  denote by  $iC$  the subcategory of isomorphisms in  $C$ ; equivalently,  $iC$  is the maximal subgroupoid in  $C$ . If  $\mathcal{C}$  is an algebroid stack then the stack associated to the substack of isomorphisms  $i\mathcal{C}$  is a gerbe.

For an algebra  $K$  we denote by  $K^+$  the linear category with a single object whose endomorphism algebra is  $K$ . For a sheaf of algebras  $\mathcal{K}$  on  $X$  we denote by  $\mathcal{K}^+$  the prestack in linear categories given by  $U \mapsto \mathcal{K}(U)^+$ . Let  $\widetilde{\mathcal{K}}^+$  denote the associated stack. Then,  $\widetilde{\mathcal{K}}^+$  is an algebroid stack equivalent to the stack of locally free  $\mathcal{K}^{\text{op}}$ -modules of rank one.

By a *twisted form of  $\mathcal{K}$*  we mean an algebroid stack locally equivalent to  $\widetilde{\mathcal{K}}^+$ . It is easy to see that the equivalence classes of twisted forms of  $\mathcal{K}$  are in bijective correspondence with  $H^2(X; \mathbf{Z}(\mathcal{K})^\times)$ , where  $\mathbf{Z}(\mathcal{K})$  denotes the center of  $\mathcal{K}$ .

**2.5. Twisted forms of  $\mathcal{O}$ .** Twisted forms of  $\mathcal{O}_X$  are in bijective correspondence with  $\mathcal{O}_X^\times$ -gerbes: if  $\mathcal{S}$  is a twisted form of  $\mathcal{O}_X$ , the corresponding gerbe is the substack  $i\mathcal{S}$  of isomorphisms in  $\mathcal{S}$ . We shall not make a distinction between the two notions.

The equivalence classes of twisted forms of  $\mathcal{O}_X$  are in bijective correspondence with  $H^2(X; \mathcal{O}_X^\times)$ . The composition

$$\mathcal{O}_X^\times \rightarrow \mathcal{O}_X^\times / \mathbb{C}^\times \xrightarrow{\log} \mathcal{O}_X / \mathbb{C} \xrightarrow{j^\infty} \text{DR}(\mathcal{J}_X / \mathcal{O}_X)$$

induces the map  $H^2(X; \mathcal{O}_X^\times) \rightarrow H^2(X; \text{DR}(\mathcal{J}_X / \mathcal{O}_X)) \cong H^2(\Gamma(X; \mathcal{A}_X^\bullet \otimes \mathcal{J}_X / \mathcal{O}_X), \nabla^{\text{can}})$ . We denote by  $[\mathcal{S}]$  the image in the latter space of the class of  $\mathcal{S}$ .

The short exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{1} \mathcal{J}_X \rightarrow \mathcal{J}_X / \mathcal{O}_X \rightarrow 0$$

gives rise to the short exact sequence of complexes

$$0 \rightarrow \Gamma(X; \mathcal{A}_X^\bullet) \rightarrow \Gamma(X; \text{DR}(\mathcal{J}_X)) \rightarrow \Gamma(X; \text{DR}(\mathcal{J}_X / \mathcal{O}_X)) \rightarrow 0,$$

hence to the map (connecting homomorphism)  $H^2(X; \text{DR}(\mathcal{J}_X / \mathcal{O}_X)) \rightarrow H_{dR}^3(X)$ . Namely, if  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  maps to  $\overline{B} \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X)$  which represents  $[\mathcal{S}]$ , then there exists a unique  $H \in \Gamma(X; \mathcal{A}^3)$  such that  $\nabla^{\text{can}} B = \text{DR}(\mathbf{1})(H)$ . The form  $H$  is closed and represents the image of the class of  $\overline{B}$  under the connecting homomorphism.

*Notation.* We denote by  $[\mathcal{S}]_{dR}$  the image of  $[\mathcal{S}]$  under the map

$$H^2(X; \text{DR}(\mathcal{J}_X / \mathcal{O}_X)) \rightarrow H_{dR}^3(X).$$

### 3. DEFORMATIONS OF ALGEBROID STACKS

**3.1. Deformations of linear stacks.** Here we describe the notion of 2-groupoid of deformations of an algebroid stack. We follow [2] and refer the reader to that paper for all the proofs and additional details.

For an  $R$ -linear category  $\mathcal{C}$  and homomorphism of algebras  $R \rightarrow S$  we denote by  $\mathcal{C} \otimes_R S$  the category with the same objects as  $\mathcal{C}$  and morphisms defined by  $\text{Hom}_{\mathcal{C} \otimes_R S}(A, B) = \text{Hom}_{\mathcal{C}}(A, B) \otimes_R S$ .

For a prestack  $\mathcal{C}$  in  $R$ -linear categories we denote by  $\mathcal{C} \otimes_R S$  the prestack associated to the fibered category  $U \mapsto \mathcal{C}(U) \otimes_R S$ .

**Lemma 3.1** ([2], Lemma 4.13). *Suppose that  $\mathcal{A}$  is a sheaf of  $R$ -algebras and  $\mathcal{C}$  is an  $R$ -algebroid stack. Then  $\widetilde{\mathcal{C} \otimes_R S}$  is an algebroid stack.*

Suppose now that  $\mathcal{C}$  is a stack in  $k$ -linear categories on  $X$  and  $R$  is a commutative Artin  $k$ -algebra. We denote by  $\text{Def}(\mathcal{C})(R)$  the 2-category with

- objects: pairs  $(\mathcal{B}, \varpi)$ , where  $\mathcal{B}$  is a stack in  $R$ -linear categories flat over  $R$  and  $\varpi : \widetilde{\mathcal{B} \otimes_R k} \rightarrow \mathcal{C}$  is an equivalence of stacks in  $k$ -linear categories
- 1-morphisms: a 1-morphism  $(\mathcal{B}^{(1)}, \varpi^{(1)}) \rightarrow (\mathcal{B}^{(2)}, \varpi^{(2)})$  is a pair  $(F, \theta)$  where  $F : \mathcal{B}^{(1)} \rightarrow \mathcal{B}^{(2)}$  is a  $R$ -linear functor and  $\theta : \varpi^{(2)} \circ (F \otimes_R k) \rightarrow \varpi^{(1)}$  is an isomorphism of functors
- 2-morphisms: a 2-morphism  $(F', \theta') \rightarrow (F'', \theta'')$  is a morphism of  $R$ -linear functors  $\kappa : F' \rightarrow F''$  such that  $\theta'' \circ (\text{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$

The 2-category  $\text{Def}(\mathcal{C})(R)$  is a 2-groupoid.

Let  $\mathcal{B}$  be a prestack on  $X$  in  $R$ -linear categories. We say that  $\mathcal{B}$  is *flat* if for any  $U \subseteq X$ ,  $A, B \in \mathcal{B}(U)$  the sheaf  $\underline{\text{Hom}}_{\mathcal{B}}(A, B)$  is flat (as a sheaf of  $R$ -modules).

**Lemma 3.2** ([2], Lemma 6.2). *Suppose that  $\mathcal{B}$  is a flat  $R$ -linear stack on  $X$  such that  $\widetilde{\mathcal{B} \otimes_R k}$  is an algebroid stack. Then  $\mathcal{B}$  is an algebroid stack.*

**3.2. Deformations of twisted forms of  $\mathcal{O}$ .** Suppose that  $\mathcal{S}$  is a twisted form of  $\mathcal{O}_X$ . We will now describe the DGLA controlling the deformations of  $\mathcal{S}$ .

The complex  $\Gamma(X; \text{DR}(C^\bullet(\mathcal{J}_X))) = (\Gamma(X; \mathcal{A}_X^\bullet \otimes C^\bullet(\mathcal{J}_X)), \nabla^{\text{can}} + \delta)$  is a differential graded brace algebra in a canonical way. The abelian Lie algebra  $\mathcal{J}_X = C^0(\mathcal{J}_X)$  acts on the brace algebra  $C^\bullet(\mathcal{J}_X)$  by derivations of degree  $-1$  by Gerstenhaber bracket. The above action factors through an action of  $\mathcal{J}_X/\mathcal{O}_X$ . Therefore, the abelian Lie algebra

$\Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X)$  acts on the brace algebra  $\mathcal{A}_X^\bullet \otimes C^\bullet(\mathcal{J}_X)$  by derivations of degree +1. Following longstanding tradition, the action of an element  $a$  is denoted by  $i_a$ .

Due to commutativity of  $\mathcal{J}_X$ , for any  $\omega \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X)$  the operation  $\iota_\omega$  commutes with the Hochschild differential  $\delta$ . If, moreover,  $\omega$  satisfies  $\nabla^{\text{can}} \omega = 0$ , then  $\nabla^{\text{can}} + \delta + i_\omega$  is a square-zero derivation of degree one of the brace structure. We refer to the complex

$$\Gamma(X; \text{DR}(C^\bullet(\mathcal{J}_X))_\omega := (\Gamma(X; \mathcal{A}_X^\bullet \otimes C^\bullet(\mathcal{J}_X)), \nabla^{\text{can}} + \delta + i_\omega)$$

as the  $\omega$ -twist of  $\Gamma(X; \text{DR}(C^\bullet(\mathcal{J}_X)))$ .

Let

$$\mathfrak{g}_{\text{DR}}(\mathcal{J})_\omega := \Gamma(X; \text{DR}(C^\bullet(\mathcal{J}_X))[1])_\omega$$

regarded as a DGLA. The following theorem is proved in [2] (Theorem 1 of loc. cit.):

**Theorem 3.3.** *For any Artin algebra  $R$  with maximal ideal  $\mathfrak{m}_R$  there is an equivalence of 2-groupoids*

$$\text{MC}^2(\mathfrak{g}_{\text{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}_R) \cong \text{Def}(\mathcal{S})(R)$$

*natural in  $R$ .*

#### 4. FORMALITY

We give a synopsis of the results of [5] in the notations of loc. cit. Let  $k$  be a field of characteristic zero. For a  $k$ -cooperad  $\mathcal{C}$  and a complex of  $k$ -vector spaces  $V$  we denote by  $\mathbb{F}_{\mathcal{C}}(V)$  the cofree  $\mathcal{C}$ -coalgebra on  $V$ .

We denote by  $\mathbf{e}_2$  the operad governing Gerstenhaber algebras. The latter is Koszul, and we denote by  $\mathbf{e}_2^\vee$  the dual cooperad.

For an associative  $k$ -algebra  $A$  the Hochschild complex  $C^\bullet(A)$  has a canonical structure of a brace algebra, hence a structure of homotopy  $\mathbf{e}_2$ -algebra. The latter structure is encoded in a differential (i.e. a coderivation of degree one and square zero)  $M: \mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)) \rightarrow \mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A))[1]$ .

Suppose from now on that  $A$  is regular commutative algebra over a field of characteristic zero (the regularity assumption is not needed for the constructions). Let  $V^\bullet(A) = \text{Sym}_A^\bullet(\text{Der}(A)[-1])$  viewed as a complex with trivial differential. In this capacity  $V^\bullet(A)$  has a canonical structure of an  $\mathbf{e}_2$ -algebra which gives rise to the differential  $d_{V^\bullet(A)}$  on  $\mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A))$ ; we have:  $\mathbb{B}_{\mathbf{e}_2^\vee}(V^\bullet(A)) = (\mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A)), d_{V^\bullet(A)})$  (see [5], Theorem 1 for notations).

In addition, the authors introduce a sub- $\mathbf{e}_2^\vee$ -coalgebra  $\Xi(A)$  of both  $\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A))$  and  $\mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A))$ . We denote by  $\sigma: \Xi(A) \rightarrow \mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A))$  and  $\iota: \Xi(A) \rightarrow \mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A))$  respective inclusions and identify  $\Xi(A)$

with its image under the latter one. By [5], Proposition 7 the differential  $d_{V^\bullet(A)}$  preserves  $\Xi(A)$ ; we denote by  $d_{V^\bullet(A)}$  its restriction to  $\Xi(A)$ . By Theorem 3, loc. cit. the inclusion  $\sigma$  is a morphism of complexes. Hence, we have the following diagram in the category of differential graded  $\mathbf{e}_2^\vee$ -coalgebras:

$$(4.0.1) \quad (\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M) \xleftarrow{\sigma} (\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\iota} \mathbf{B}_{\mathbf{e}_2^\vee}(V^\bullet(A))$$

Applying the functor  $\Omega_{\mathbf{e}_2}$  (adjoint to the functor  $\mathbf{B}_{\mathbf{e}_2^\vee}$ , see [5], Theorem 1) to (4.0.1) we obtain the diagram

$$(4.0.2) \quad \Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M) \xleftarrow{\Omega_{\mathbf{e}_2}(\sigma)} \Omega_{\mathbf{e}_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\Omega_{\mathbf{e}_2}(\iota)} \Omega_{\mathbf{e}_2}(\mathbf{B}_{\mathbf{e}_2^\vee}(V^\bullet(A)))$$

of differential graded  $\mathbf{e}_2$ -algebras. Let  $\nu = \eta_{\mathbf{e}_2} \circ \Omega_{\mathbf{e}_2}(\iota)$ , where  $\eta_{\mathbf{e}_2} : \Omega_{\mathbf{e}_2}(\mathbf{B}_{\mathbf{e}_2^\vee}(V^\bullet(A))) \rightarrow V^\bullet(A)$  is the counit of adjunction. Thus, we have the diagram

$$(4.0.3) \quad \Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M) \xleftarrow{\Omega_{\mathbf{e}_2}(\sigma)} \Omega_{\mathbf{e}_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\nu} V^\bullet(A)$$

of differential graded  $\mathbf{e}_2$ -algebras.

**Theorem 4.1** ([5], Theorem 4). *The maps  $\Omega_{\mathbf{e}_2}(\sigma)$  and  $\nu$  are quasi-isomorphisms.*

Additionally, concerning the DGLA structures relevant to applications to deformation theory, deduced from respective  $\mathbf{e}_2$ -algebra structures we have the following result.

**Theorem 4.2** ([5], Theorem 2). *The DGLA  $\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M)[1]$  and  $C^\bullet(A)[1]$  are canonically  $L_\infty$ -quasi-isomorphic.*

**Corollary 4.3** (Formality). *The DGLA  $C^\bullet(A)[1]$  and  $V^\bullet(A)[1]$  are  $L_\infty$ -quasi-isomorphic.*

**4.1. Some (super-)symmetries.** For applications to deformation theory of algebroid stacks we will need certain equivariance properties of the maps described in 4.

For  $a \in A$  let  $i_a : C^\bullet(A) \rightarrow C^\bullet(A)[-1]$  denote the adjoint action (in the sense of the Gerstenhaber bracket and the identification  $A = C^0(A)$ ). It is given by the formula

$$i_a D(a_1, \dots, a_n) = \sum_{i=0}^n (-1)^k D(a_1, \dots, a_i, a, a_{k+1}, \dots, a_n).$$

The operation  $i_a$  extends uniquely to a coderivation of  $\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A))$ ; we denote this extension by  $i_a$  as well. Furthermore, the subcoalgebra  $\Xi(A)$  is preserved by  $i_a$ .



Since the operation  $i_a$  is a derivation of the cup product as well as of all of the brace operations on  $C^\bullet(A)$  and the homotopy- $\mathbf{e}_2$ -algebra structure on  $C^\bullet(A)$  given in terms of the cup product and the brace operations it follows that  $i_a$  anti-commutes with the differential  $M$ . Hence, the coderivation  $i_a$  induces a derivation of the differential graded  $\mathbf{e}_2$ -algebra  $\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M)$  which will be denoted by  $i_a$  as well. For the same reason the DGLA  $\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A)), M)[1]$  and  $C^\bullet(A)[1]$  are quasi-isomorphic in a way which commutes with the respective operations  $i_a$ .

On the other hand, let  $i_a: V^\bullet(A) \rightarrow V^\bullet(A)[-1]$  denote the adjoint action in the sense of the Schouten bracket and the identification  $A = V^0(A)$ . The operation  $i_a$  extends uniquely to a coderivation of  $\mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A))$  which anticommutes with the differential  $d_{V^\bullet(A)}$  because  $i_a$  is a derivation of the  $\mathbf{e}_2$ -algebra structure on  $V^\bullet(A)$ . We denote this coderivation as well as its unique extension to a derivation of the differential graded  $\mathbf{e}_2$ -algebra  $\Omega_{\mathbf{e}_2}(\mathbb{B}_{\mathbf{e}_2^\vee}(V^\bullet(A)))$  by  $i_a$ . The counit map  $\eta_{\mathbf{e}_2}: \Omega_{\mathbf{e}_2}(\mathbb{B}_{\mathbf{e}_2^\vee}(V^\bullet(A))) \rightarrow V^\bullet(A)$  commutes with respective operations  $i_a$ .

The subcoalgebra  $\Xi(A)$  of  $\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(A))$  and  $\mathbb{F}_{\mathbf{e}_2^\vee}(V^\bullet(A))$  is preserved by the respective operations  $i_a$ . Moreover, the restrictions of the two operations to  $\Xi(A)$  coincide, i.e. the maps in (4.0.1) commute with  $i_a$  and, therefore, so do the maps in (4.0.2) and (4.0.3).

**4.2. Deformations of  $\mathcal{O}$  and Kontsevich formality.** Suppose that  $X$  is a manifold. Let  $\mathcal{O}_X$  (respectively,  $\mathcal{T}_X$ ) denote the structure sheaf (respectively, the sheaf of vector fields). The construction of the diagram localizes on  $X$  yielding the diagram of sheaves of differential graded  $\mathbf{e}_2$ -algebras

$$(4.2.1) \quad \Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{O}_X)), M) \xleftarrow{\Omega_{\mathbf{e}_2}(\sigma)} \Omega_{\mathbf{e}_2}(\Xi(\mathcal{O}_X), d_{V^\bullet(\mathcal{O}_X)}) \xrightarrow{\nu} V^\bullet(\mathcal{O}_X),$$

where  $C^\bullet(\mathcal{O}_X)$  denotes the sheaf of multidifferential operators and  $V^\bullet(\mathcal{O}_X) := \text{Sym}_{\mathcal{O}_X}^\bullet(\mathcal{T}_X[-1])$  denotes the sheaf of multivector fields. Theorem 4.1 extends easily to this case stating that the morphisms  $\Omega_{\mathbf{e}_2}(\sigma)$  and  $\nu$  in (4.2.1) are quasi-isomorphisms of sheaves of differential graded  $\mathbf{e}_2$ -algebras.

## 5. FORMALITY FOR THE ALGEBROID HOCHSCHILD COMPLEX

**5.1. A version of [5] for jets.** Let  $C^\bullet(\mathcal{J}_X)$  denote sheaf of continuous (with respect to the adic topology)  $\mathcal{O}_X$ -multilinear Hochschild cochains on  $\mathcal{J}_X$ . Let  $V^\bullet(\mathcal{J}_X) = \text{Sym}_{\mathcal{O}_X}^\bullet(\text{Der}_{\mathcal{O}_X}^{\text{cont}}(\mathcal{J}_X)[-1])$ .

Working now in the category of graded  $\mathcal{O}_X$ -modules we have the diagram

(5.1.1)

$$\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M) \xleftarrow{\Omega_{\mathbf{e}_2}(\sigma)} \Omega_{\mathbf{e}_2}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}) \xrightarrow{\nu} V^\bullet(\mathcal{J}_X)$$

of sheaves of differential graded  $\mathcal{O}_X$ - $\mathbf{e}_2$ -algebras. Theorem 4.1 extends easily to this situation: the morphisms  $\Omega_{\mathbf{e}_2}(\sigma)$  and  $\nu$  in (5.1.1) are quasi-isomorphisms. The sheaves of DGLA  $\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)[1]$  and  $C^\bullet(\mathcal{J}_X)[1]$  are canonically  $L_\infty$ -quasi-isomorphic.

The canonical flat connection  $\nabla^{can}$  on  $\mathcal{J}_X$  induces a flat connection which we denote  $\nabla^{can}$  as well on each of the objects in the diagram (5.1.1). Moreover, the maps  $\Omega_{\mathbf{e}_2}(\sigma)$  and  $\nu$  are flat with respect to  $\nabla^{can}$  hence induce the maps of respective de Rham complexes

$$(5.1.2) \quad \mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)) \xleftarrow{\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))} \mathrm{DR}(\Omega_{\mathbf{e}_2}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)})) \xrightarrow{\mathrm{DR}(\nu)} \mathrm{DR}(V^\bullet(\mathcal{J}_X))$$

where, for  $(K^\bullet, d)$  one of the objects in (5.1.1) we denote by  $\mathrm{DR}(K^\bullet, d)$  the total complex of the double complex  $(\mathcal{A}_X^\bullet \otimes K^\bullet, d, \nabla^{can})$ . All objects in the diagram (5.1.2) have canonical structures of differential graded  $\mathbf{e}_2$ -algebras and the maps are morphisms thereof.

The DGLA  $\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)[1]$  and  $C^\bullet(\mathcal{J}_X)[1]$  are canonically  $L_\infty$ -quasi-isomorphic in a way compatible with  $\nabla^{can}$ . Hence, the DGLA  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])$  and  $\mathrm{DR}(C^\bullet(\mathcal{J}_X)[1])$  are canonically  $L_\infty$ -quasi-isomorphic.

**5.2. A version of [5] for jets with a twist.** Suppose that  $\omega \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X / \mathcal{O}_X)$  satisfies  $\nabla^{can}\omega = 0$ .

For each of the objects in (5.1.2) we denote by  $i_\omega$  the operation which is induced by the one described in 4.1 and the wedge product on  $\mathcal{A}_X^\bullet$ . Thus, for each differential graded  $\mathbf{e}_2$ -algebra  $(N^\bullet, d)$  in (5.1.2) we have a derivation of degree one and square zero  $i_\omega$  which anticommutes with  $d$  and we denote by  $(N^\bullet, d)_\omega$  the  $\omega$ -twist of  $(N^\bullet, d)$ , i.e. the differential graded  $\mathbf{e}_2$ -algebra  $(N^\bullet, d + i_\omega)$ . Since the morphisms in (5.1.2) commute with the respective operations  $i_\omega$ , they give rise to morphisms of respective  $\omega$ -twists

$$(5.2.1) \quad \mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M))_\omega \xleftarrow{\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))} \mathrm{DR}(\Omega_{\mathbf{e}_2}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}))_\omega \xrightarrow{\mathrm{DR}(\nu)} \mathrm{DR}(V^\bullet(\mathcal{J}_X))_\omega.$$

Let  $F_\bullet \mathcal{A}_X^\bullet$  denote the stupid filtration:  $F_i \mathcal{A}_X^\bullet = \mathcal{A}_X^{\geq -i}$ . The filtration  $F_\bullet \mathcal{A}_X^\bullet$  induces a filtration denoted  $F_\bullet \mathrm{DR}(K^\bullet, d)_\omega$  for each object  $(K^\bullet, d)$

of (5.1.1) defined by  $F_i \mathrm{DR}(K^\bullet, d)_\omega = F_i \mathcal{A}_X^\bullet \otimes K^\bullet$ . As is easy to see, the associated graded complex is given by

$$(5.2.2) \quad Gr_{-p} \mathrm{DR}(K^\bullet, d)_\omega = (\mathcal{A}_X^p \otimes K^\bullet, \mathrm{Id} \otimes d).$$

It is clear that the morphisms  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  and  $\mathrm{DR}(\nu)$  are filtered with respect to  $F_\bullet$ .

**Theorem 5.1.** *The morphisms in (5.2.1) are filtered quasi-isomorphisms, i.e. the maps  $Gr_i \mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  and  $Gr_i \mathrm{DR}(\nu)$  are quasi-isomorphisms for all  $i \in \mathbb{Z}$ .*

*Proof.* We consider the case of  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  leaving  $Gr_i \mathrm{DR}(\nu)$  to the reader.

The map  $Gr_{-p} \mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  induced by  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  on the respective associated graded objects in degree  $-p$  is equal to the map of complexes

$$(5.2.3) \quad \mathrm{Id} \otimes \Omega_{\mathbf{e}_2}(\sigma): \mathcal{A}_X^p \otimes \Omega_{\mathbf{e}_2}(\Xi(\mathcal{J}_X), d_{V^\bullet(\mathcal{J}_X)}) \rightarrow \mathcal{A}_X^p \otimes \Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M).$$

The map  $\sigma$  is a quasi-isomorphism by Theorem 4.1, therefore so is  $\Omega_{\mathbf{e}_2}(\sigma)$ . Since  $\mathcal{A}_X^p$  is flat over  $\mathcal{O}_X$ , the map (5.2.3) is a quasi-isomorphism.  $\square$

**Corollary 5.2.** *The maps  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\sigma))$  and  $\mathrm{DR}(\nu)$  in (5.2.1) are quasi-isomorphisms of sheaves of differential graded  $\mathbf{e}_2$ -algebras.*

Additionally, the DGLA  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])$  and  $\mathrm{DR}(C^\bullet(\mathcal{J}_X)[1])$  are canonically  $L_\infty$ -quasi-isomorphic in a way which commutes with the respective operations  $i_\omega$  which implies that the respective  $\omega$ -twists  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M)[1])_\omega$  and  $\mathrm{DR}(C^\bullet(\mathcal{J}_X)[1])_\omega$  are canonically  $L_\infty$ -quasi-isomorphic.

**5.3.  $L_\infty$ -structures on multivectors.** The canonical pairing  $\langle \cdot, \cdot \rangle: \mathcal{A}_X^1 \otimes T_X \rightarrow \mathcal{O}_X$  extends to the pairing

$$\langle \cdot, \cdot \rangle: \mathcal{A}_X^1 \otimes V^\bullet(\mathcal{O}_X) \rightarrow V^\bullet(\mathcal{O}_X)[-1]$$

For  $k \geq 1$ ,  $\omega = \alpha_1 \wedge \dots \wedge \alpha_k$ ,  $\alpha_i \in \mathcal{A}_X^1$ ,  $i = 1, \dots, k$ , let

$$\Phi(\omega): \mathrm{Sym}^k V^\bullet(\mathcal{O}_X)[2] \rightarrow V^\bullet(\mathcal{O}_X)[k]$$

denote the map given by the formula

$$\begin{aligned} \Phi(\omega)(\pi_1, \dots, \pi_k) &= (-1)^{(k-1)(|\pi_1|-1) + \dots + 2(|\pi_{k-3}|-1) + (|\pi_{k-2}|-1)} \times \\ &\quad \sum_{\sigma} \mathrm{sgn}(\sigma) \langle \alpha_1, \pi_{\sigma(1)} \rangle \wedge \dots \wedge \langle \alpha_k, \pi_{\sigma(k)} \rangle, \end{aligned}$$

where  $|\pi| = l$  for  $\pi \in V^l(\mathcal{O}_X)$ . For  $\alpha \in \mathcal{O}_X$  let  $\Phi(\alpha) = \alpha \in V^0(\mathcal{O}_X)$ .

Recall that a graded vector space  $W$  gives rise to the graded Lie algebra  $\mathbf{Der}(\mathrm{coComm}(W[1]))$ . An element  $\gamma \in \mathbf{Der}(\mathrm{coComm}(W[1]))$  of degree one which satisfies  $[\gamma, \gamma] = 0$  defines a structure of an  $L_\infty$ -algebra on  $W$ . Such a  $\gamma$  determines a differential  $\partial_\gamma := [\gamma, \cdot]$  on  $\mathbf{Der}(\mathrm{coComm}(W[1]))$ , such that  $(\mathbf{Der}(\mathrm{coComm}(W[1])), \partial_\gamma)$  is a differential graded Lie algebra. If  $\mathfrak{g}$  is a graded Lie algebra and  $\gamma$  is the element of  $\mathbf{Der}(\mathrm{coComm}(\mathfrak{g}[1]))$  corresponding to the bracket on  $\mathfrak{g}$ , then  $(\mathbf{Der}(\mathrm{coComm}(\mathfrak{g}[1])), \partial_\gamma)$  is equal to the shifted Chevalley cochain complex  $C^\bullet(\mathfrak{g}; \mathfrak{g})[1]$ .

In what follows we consider the (shifted) de Rham complex  $\mathcal{A}_X^\bullet[2]$  as a differential graded Lie algebra with the trivial bracket.

**Lemma 5.3.** *The map  $\omega \mapsto \Phi(\omega)$  defines a morphism of sheaves of differential graded Lie algebras*

$$(5.3.1) \quad \Phi: \mathcal{A}_X^\bullet[2] \rightarrow C^\bullet(V^\bullet(\mathcal{O}_X)[1]; V^\bullet(\mathcal{O}_X)[1])[1].$$

*Proof.* Recall the explicit formulas for the Schouten bracket. If  $f$  and  $g$  are functions and  $X_i, Y_j$  are vector fields, then

$$\begin{aligned} [fX_1 \dots X_k, gY_1 \dots Y_l] &= \sum_i (-1)^{k-i} fX_k(g)X_1 \dots \widehat{X_i} \dots X_k Y_1 \dots Y_l + \\ &\quad \sum_j (-1)^j Y_j(f)gX_1 \dots X_k Y_1 \dots \widehat{Y_j} \dots Y_l + \\ &\quad \sum_{i,j} (-1)^{i+j} fgX_1 \dots \widehat{X_i} \dots X_k Y_1 \dots \widehat{Y_j} \dots Y_l \end{aligned}$$

Note that for a one-form  $\omega$  and for vector fields  $X$  and  $Y$

$$(5.3.2) \quad \langle \omega, [X, Y] \rangle - \langle [\omega, X], Y \rangle - \langle X, [\omega, Y] \rangle = \Phi(d\omega)(X, Y)$$

From the two formulas above we deduce by an explicit computation that

$$\langle \omega, [\pi, \rho] \rangle - \langle [\omega, \pi], \rho \rangle - (-1)^{|\pi|-1} \langle \pi, [\omega, \rho] \rangle = (-1)^{|\pi|-1} \Phi(d\omega)(\pi, \rho)$$

Note that Lie algebra cochains are invariant under the symmetric group acting by permutations multiplied by signs that are computed by the following rule: a permutation of  $\pi_i$  and  $\pi_j$  contributes a factor  $(-1)^{|\pi_i||\pi_j|}$ . We use the explicit formula for the bracket on the Lie algebra complex.

$$[\Phi, \Psi] = \Phi \circ \Psi - (-1)^{|\Phi||\Psi|} \Psi \circ \Phi$$

$$(\Phi \circ \Psi)(\pi_1, \dots, \pi_{k+l-1}) = \sum_{I, J} \epsilon(I, J) \Phi(\Psi(\pi_{i_1}, \dots, \pi_{i_k}), \pi_{j_1}, \dots, \pi_{j_{l-1}})$$

Here  $I = \{i_1, \dots, i_k\}$ ;  $J = \{j_1, \dots, j_{l-1}\}$ ;  $i_1 < \dots < i_k$ ;  $j_1 < \dots < j_{l-1}$ ;  $I \amalg J = \{1, \dots, k+l-1\}$ ; the sign  $\epsilon(I, J)$  is computed by the same sign rule as above. The differential is given by the formula

$$\partial\Phi = [m, \Phi]$$

where  $m(\pi, \rho) = (-1)^{|\pi|-1}[\pi, \rho]$ . Let  $\alpha = \alpha_1 \dots \alpha_k$  and  $\beta = \beta_1 \dots \beta_l$ . We see from the above that both cochains  $\Phi(\alpha) \circ \Phi(\beta)$  and  $\Phi(\beta) \circ \Phi(\alpha)$  are antisymmetrizations with respect to  $\alpha_i$  and  $\beta_j$  of the sums

$$\sum_{I, J, p} \pm \langle \alpha_1 \beta_1, \pi_p \rangle \langle \alpha_2, \pi_{i_1} \rangle \dots \langle \alpha_k, \pi_{i_{k-1}} \rangle \langle \beta_2, \pi_{j_1} \rangle \dots \langle \beta_l, \pi_{j_{l-1}} \rangle$$

over all partitions  $\{1, \dots, k+l-1\} = I \amalg J \amalg \{p\}$  where  $i_1 < \dots < i_{k-1}$  and  $j_1 < \dots < j_{l-1}$ ; here  $\langle \alpha\beta, \pi \rangle = \langle \alpha, \langle \beta, \pi \rangle \rangle$ . After checking the signs, we conclude that  $[\Phi(\alpha), \Phi(\beta)] = 0$ . Also, from the definition of the differential, we see that  $\partial\Phi(\alpha)(\pi_1, \dots, \pi_{k+1})$  is the antisymmetrizations with respect to  $\alpha_i$  and  $\beta_j$  of the sum

$$\sum_{i < j} \pm (\langle \alpha_1, [\pi_i, \pi_j] \rangle - \langle [\alpha_1, \pi_i], \pi_j \rangle - (-1)^{|\pi_i|-1} [\pi_i, \langle \alpha_1, \pi_j \rangle]) \cdot \langle \alpha_2, \pi_1 \rangle \dots \langle \alpha_i, \pi_{i-1} \rangle \langle \alpha_{i+1}, \pi_{i+1} \rangle \dots \langle \alpha_{j-1}, \pi_{j-1} \rangle \langle \alpha_j, \pi_{j+1} \rangle \langle \alpha_k, \pi_{k+1} \rangle$$

We conclude from this and (5.3.2) that  $\partial\Phi(\alpha) = \Phi(d\alpha)$ .  $\square$

Thus, according to Lemma 5.3, a closed 3-form  $H$  on  $X$  gives rise to a Maurer-Cartan element  $\Phi(H)$  in  $\Gamma(X; C^\bullet(V^\bullet(\mathcal{O}_X)[1]; V^\bullet(\mathcal{O}_X)[1])[1])$ , hence a structure of an  $L_\infty$ -algebra on  $V^\bullet(\mathcal{O}_X)[1]$  which has the trivial differential (the unary operation), the binary operation equal to the Schouten-Nijenhuis bracket, the ternary operation given by  $\Phi(H)$ , and all higher operations equal to zero. Moreover, cohomologous closed 3-forms give rise to gauge equivalent Maurer-Cartan elements, hence to  $L_\infty$ -isomorphic  $L_\infty$ -structures.

*Notation.* For a closed 3-form  $H$  on  $X$  we denote the corresponding  $L_\infty$ -algebra structure on  $V^\bullet(\mathcal{O}_X)[1]$  by  $V^\bullet(\mathcal{O}_X)[1]_H$ . Let

$$\mathfrak{s}(\mathcal{O}_X)_H := \Gamma(X; V^\bullet(\mathcal{O}_X)[1])_H.$$

**5.4.  $L_\infty$ -structures on multivectors via formal geometry.** In order to relate the results of 5.2 with those of 5.3 we consider the analog of the latter for jets.

Let  $\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^k := \mathcal{J}_X(\mathcal{A}_X^k)$ , the sheaf of jets of differential  $k$ -forms on  $X$ . Let  $\widehat{d}_{\mathbb{R}}$  denote the  $(\mathcal{O}_X$ -linear) differential in  $\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet$  induced by the de Rham differential in  $\mathcal{A}_X^\bullet$ . The differential  $\widehat{d}_{\mathbb{R}}$  is horizontal with respect to the canonical flat connection  $\nabla^{can}$  on  $\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet$ , hence we have

the double complex  $(\mathcal{A}_X^\bullet \otimes \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet, \nabla^{can}, \text{Id} \otimes \widehat{d}_R)$  whose total complex is denoted  $\text{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet)$ .

Let  $\mathbf{1}: \mathcal{O}_X \rightarrow \mathcal{J}_X$  denote the unit map (not to be confused with the map  $j^\infty$ ); it is an isomorphism onto the kernel of  $\widehat{d}_R: \mathcal{J}_X \rightarrow \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^1$  and therefore defines the morphism of complexes  $\mathbf{1}: \mathcal{O}_X \rightarrow \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet$  which is a quasi-isomorphism. The map  $\mathbf{1}$  is horizontal with respect to the canonical flat connections on  $\mathcal{O}_X$  and  $\mathcal{J}_X$  (respectively,  $\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet$ ), therefore we have the induced map of respective de Rham complexes  $\text{DR}(\mathbf{1}): \mathcal{A}_X^\bullet \rightarrow \text{DR}(\mathcal{J}_X)$  (respectively,  $\text{DR}(\mathbf{1}): \mathcal{A}_X^\bullet \rightarrow \text{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet)$ , a quasi-isomorphism).

Let  $C^\bullet(\mathfrak{g}(\mathcal{J}_X); \mathfrak{g}(\mathcal{J}_X))$  denote the complex of continuous  $\mathcal{O}_X$ -multilinear cochains. The map of differential graded Lie algebras

$$(5.4.1) \quad \widehat{\Phi}: \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet[2] \rightarrow C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1]$$

defined in the same way as (5.3.1) is horizontal with respect to the canonical flat connection  $\nabla^{can}$  and induces the map

$$(5.4.2) \quad \text{DR}(\widehat{\Phi}): \text{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet[2]) \rightarrow \text{DR}((C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1])$$

There is a canonical morphism of sheaves of differential graded Lie algebras

$$(5.4.3) \quad \text{DR}(C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1]) \rightarrow C^\bullet(\text{DR}(V^\bullet(\mathcal{J}_X)[1]); \text{DR}(V^\bullet(\mathcal{J}_X)[1]))[1]$$

Therefore, a degree three cocycle in  $\Gamma(X; \text{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet))$  determines an  $L_\infty$ -structure on  $\text{DR}(V^\bullet(\mathcal{J}_X)[1])$  and cohomologous cocycles determine  $L_\infty$ -isomorphic structures.

*Notation.* For a section  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  we denote by  $\overline{B}$  its image in  $\Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X)$ .

**Lemma 5.4.** *If  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  satisfies  $\nabla^{can} \overline{B} = 0$ , then*

- (1)  $\widehat{d}_R B$  is a (degree three) cocycle in  $\Gamma(X; \text{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet))$ ;
- (2) there exist a unique  $H \in \Gamma(X; \mathcal{A}_X^3)$  such that  $dH = 0$  and  $\text{DR}(\mathbf{1})(H) = \nabla^{can} \overline{B}$ .

*Proof.* For the first claim it suffices to show that  $\nabla^{can} \overline{B} = 0$ . This follows from the assumption that  $\nabla^{can} \overline{B} = 0$  and the fact that  $\widehat{d}_R: \mathcal{A}_X^\bullet \otimes \mathcal{J}_X \rightarrow \mathcal{A}_X^\bullet \otimes \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^1$  factors through  $\mathcal{A}_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X$ .

We have:  $\widehat{d}_R \nabla^{can} B = \nabla^{can} \widehat{d}_R B = 0$ . Therefore,  $\nabla^{can} B$  is in the image of  $\text{DR}(\mathbf{1}): \Gamma(X; \mathcal{A}_X^3) \rightarrow \Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X)$  which is injective, whence the existence and uniqueness of  $H$ . Since  $\text{DR}(\mathbf{1})$  is a morphism of complexes it follows that  $H$  is closed.  $\square$

Suppose that  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  satisfies  $\nabla^{can} \overline{B} = 0$ . Then, the differential graded Lie algebra  $\mathrm{DR}(\mathfrak{g}(\mathcal{J}_X))_{\overline{B}}$  (the  $\overline{B}$ -twist of  $\mathrm{DR}(\mathfrak{g}(\mathcal{J}_X))$ ) is defined. On the other hand, due to Lemma 5.4, (5.4.2) and (5.4.3),  $\widehat{d}_R B$  gives rise to an  $L_\infty$ -structure on  $\mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])$ .

**Lemma 5.5.** *The  $L_\infty$ -structure induced by  $\widehat{d}_R B$  is that of a differential graded Lie algebra equal to  $\mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$ .*

*Proof.* Left to the reader.  $\square$

*Notation.* For a 3-cocycle  $\omega \in \Gamma(X; \mathrm{DR}(\widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^\bullet))$  we will denote by  $\mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_\omega$  the  $L_\infty$ -algebra obtained from  $\omega$  via (5.4.2) and (5.4.3). Let

$$\mathfrak{s}_{\mathrm{DR}}(\mathcal{J}_X)_\omega := \Gamma(X; \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1]))_\omega.$$

*Remark 5.6.* Lemma 5.5 shows that this notation is unambiguous with reference to the previously introduced notation for the twist. In the notations introduced above,  $\widehat{d}_R B$  is the image of  $\overline{B}$  under the *injective* map  $\Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X) \rightarrow \Gamma(X; \mathcal{A}_X^2 \otimes \widehat{\Omega}_{\mathcal{J}/\mathcal{O}}^1)$  which factors  $\widehat{d}_R$  and “allows” us to “identify”  $\overline{B}$  with  $\widehat{d}_R B$ .

**Theorem 5.7.** *Suppose that  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  satisfies  $\nabla^{can} \overline{B} = 0$ . Let  $H \in \Gamma(X; \mathcal{A}_X^3)$  denote the unique 3-form such that  $\mathrm{DR}(\mathbf{1})(H) = \nabla^{can} B$  (cf. Lemma 5.4). Then, the  $L_\infty$ -algebras  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\overline{B}}$  and  $\mathfrak{s}(\mathcal{O}_X)_H$  are  $L_\infty$ -quasi-isomorphic.*

*Proof.* The map  $j^\infty: V^\bullet(\mathcal{O}_X) \rightarrow V^\bullet(\mathcal{J}_X)$  induces a quasi-isomorphism of sheaves of DGLA

$$(5.4.4) \quad j^\infty: V^\bullet(\mathcal{O}_X)[1] \rightarrow \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1]).$$

Suppose that  $H$  is a closed 3-form on  $X$ . Then, the map (5.4.4) is a quasi-isomorphism of sheaves of  $L_\infty$ -algebras

$$j^\infty: V^\bullet(\mathcal{O}_X)[1]_H \rightarrow \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathrm{DR}(\mathbf{1})(H)}.$$

Passing to global section we obtain the quasi-isomorphism of  $L_\infty$ -algebras

$$(5.4.5) \quad j^\infty: \mathfrak{s}(\mathcal{O}_X)_H \rightarrow \mathfrak{s}_{\mathrm{DR}}(\mathcal{J}_X)_{\mathrm{DR}(\mathbf{1})(H)}.$$

By assumption,  $B$  provides a homology between  $\widehat{d}_R B$  and  $\nabla^{can} B = \mathrm{DR}(\mathbf{1})(H)$ . Therefore, we have the corresponding  $L_\infty$ -quasi-isomorphism

$$(5.4.6) \quad \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathrm{DR}(\mathbf{1})(H)} \stackrel{L_\infty}{\cong} \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\widehat{d}_R B} = \mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$$

(the second equality is due to Lemma 5.5).

According to Corollary 5.2 the sheaf of DGLA  $\mathrm{DR}(V^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$  is  $L_\infty$ -quasi-isomorphic to the DGLA deduced from the differential graded  $\mathbf{e}_2$ -algebra  $\mathrm{DR}(\Omega_{\mathbf{e}_2}(\mathbb{F}_{\mathbf{e}_2^\vee}(C^\bullet(\mathcal{J}_X)), M))_{\overline{B}}$ . The latter DGLA is  $L_\infty$ -quasi-isomorphic to  $\mathrm{DR}(C^\bullet(\mathcal{J}_X)[1])_{\overline{B}}$ .

Passing to global sections we conclude that  $\mathfrak{s}_{\mathrm{DR}}(\mathcal{J}_X)_{\mathrm{DR}(\mathbf{1})(H)}$  and  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\overline{B}}$  are  $L_\infty$ -quasi-isomorphic. Together with (5.4.5) this implies the claim.  $\square$

## 6. APPLICATION TO DEFORMATION THEORY

**Theorem 6.1.** *Suppose that  $\mathcal{S}$  is a twisted form of  $\mathcal{O}_X$  (2.5). Let  $H$  be a closed 3-form on  $X$  which represents  $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$ . For any Artin algebra  $R$  with maximal ideal  $\mathfrak{m}_R$  there is an equivalence of 2-groupoids*

$$\mathrm{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R)$$

*natural in  $R$ .*

*Proof.* Since cohomologous 3-forms give rise to  $L_\infty$ -quasi-isomorphic  $L_\infty$ -algebras we may assume, possibly replacing  $H$  by another representative of  $[\mathcal{S}]_{dR}$ , that there exists  $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$  such that  $\overline{B}$  represents  $[\mathcal{S}]$  and  $\nabla^{\mathrm{can}} B = \mathrm{DR}(\mathbf{1})(H)$ . By Theorem 5.7  $\mathfrak{s}(\mathcal{O}_X)_H$  is  $L_\infty$ -quasi-isomorphic to  $\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\overline{B}}$ . By the Theorem 2.1 we have a homotopy equivalence of nerves of 2-groupoids  $\gamma_\bullet(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \gamma_\bullet(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\overline{B}} \otimes \mathfrak{m}_R)$ . Therefore, there are equivalences

$$\mathrm{MC}^2(\mathfrak{s}(\mathcal{O}_X)_H \otimes \mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_{\overline{B}} \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R),$$

the second one being that of Theorem 3.3.  $\square$

*Remark 6.2.* In particular, the isomorphism classes of formal deformations of  $\mathcal{S}$  are in a bijective correspondence with equivalence classes of Maurer-Cartan elements of the  $L_\infty$ -algebra  $\mathfrak{s}_{\mathrm{DR}}(\mathcal{O}_X)_H \widehat{\otimes} t \cdot \mathbb{C}[[t]]$ . These are the formal *twisted Poisson structures* in the terminology of [13], i.e. the formal series  $\pi = \sum_{k=1}^\infty t^k \pi_k$ ,  $\pi_k \in \Gamma(X; \bigwedge^2 \mathcal{T}_X)$ , satisfying the equation

$$[\pi, \pi] = \Phi(H)(\pi, \pi, \pi).$$

A construction of an algebroid stack associated to a twisted Poisson structure was proposed by P. Ševera in [12].

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY

*E-mail address:* paul.bressler@gmail.com

DEPARTMENT OF MATHEMATICS, UCB 395, UNIVERSITY OF COLORADO, BOULDER, CO 80309-0395, USA

*E-mail address:* Alexander.Gorokhovsky@colorado.edu

DEPARTMENT OF MATHEMATICS, COPENHAGEN UNIVERSITY, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK

*E-mail address:* rnest@math.ku.dk

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2730, USA

*E-mail address:* tsygan@math.northwestern.edu